Quasi-analysis
A method to order non-homogeneus sets by means of the theory of relations

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Abbreviations: Al[1] : Analysis into components,
Ct : Component,
QAl : Quasi-analysis,
QCt : Quasi-component,
SC : similarity circle.[2]

1. *The task of quasi-analysis*

The nature of individual objects (henceforth "elements") of any domain (henceforth "set" of elements) can be indicated by means of two different methods. The first method indicates for any individual element the characteristics that belong to it or the components (Ct) that it is composed of. We call this method analysis into components (Al). This name is also appropriate for those indications of characteristics that do not analyse the object but its concept, considered as the totality of the characteristics of the object. The second method indicates the relations that hold between elements. We call this method relational description. Although each of the two methods offers several variants, these are, however, more or less similar among themselves. At the same time, the methods are basically different. Indeed, following the first, one can make a statement about an individual element without taking into account other elements. Following the second, instead, every statement concerns only the relations of an element to one or more other elements. The two methods

[1] Carnap uses several abbreviations in the manuscript with the aim of rendering the text readable more easily. In the original manuscript, the abbreviations are chosen in accordance with the German words. In this translation, we have rendered the abbreviations consistent with the English words. This choice is motivated by the purpose of bestowing new original clarity, readability and elegance on the English version of the text.

[2] Our translation choice of "similarity circle" and "similarity neighbourhood" (6-7), standing for Carnap's original phrases "*Familienklasse*" and "*Verwandtenklasse*", is unliteral. This choice is justified by the purpose of keeping the language of *Quasizerlegung* on a homogeneous line of expression with that of the subsequent philosophical debate on quasi-analysis and similarity structure.

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could be labelled, respectively, as indication of individual properties and indication of relational properties.

> Examples of Al: description of the set of conic sections through an account of the characteristics of the individual sections; description of a curve through its coordinate equation, *i.e.*, by giving the ordinate for each point on the abscissa; description of a physical state through the values of one (or more) state variable for every position; chemical description of a given substance through its composition of chemical elements; list of historical persons with a statement of the dates of birth and death for each of them.
>
> Examples of relational description: description of a geometrical figure which consists of points and straight lines through an indication of the relations of incidence; description of a curve through its natural equation, *i.e.*, through an indication of the position of each element of the curve relative to the preceding ones; description of a physical state through spatio-temporal differential equations, *i.e.*, through the relation between the value of a state variable in some spatio-temporal point and its values in the spatio-temporal neighbourhood; description of a group of persons by means of a genealogy, *i.e.*, by giving their kinship relations.

In opposition to Al, the relational description has the advantage that it does not overstep the given domain of objects. The elements of the set to be described are, indeed, not analysed into components (Ct), whose set is generally not included in the given one. The relational description is, as it were, an "immanent approach". On the other hand, the relational description has the drawback of being ponderous in the approach to the individual elements themselves. One cannot, indeed, make a statement about an element without reference to other elements, which are again characterised only through reference to other elements, and so forth.

Now, a method will be discussed here that allows a relational description to transform a given description in such a way that retains the properties of the immanent approach and assumes the form of the analysis. Thus, a single approach to the elements is possible. This transformation is called quasi-analysis (QAl).

The simplest version of QAl considered in the following discussion can be applied everywhere, even where it seems desirable to switch to more complicated versions. Quasi-analysis starts from a relational description based on a symmetric, reflexive and non-transitive relation. Let us call this initial relation

"S" and two elements a and b, such that aSb, "similar elements", "similarity pair" or "S-pair".

If a relational description is based on a relation P whose properties are different from the above-mentioned ones (symmetry, reflexivity, intransitivity) or on many relations, P, Q, …, then one must take as basic relation S certain transformations of these other relations. One can, if necessary, apply several of the following transformations one after the other.

1) P is transitive, in particular:
   a) transitive and symmetric. Here a degenerate case occurs: all the elements are similar one to the other (with the exception of the isolated ones, i.e., the elements that are similar only to themselves). Therefore, the set is homogeneous. There are no distinguishing properties and, consequently, no possibility of order.
   b) P is transitive but not symmetric. One applies the transformation 2), whereby transitivity is also removed.
2) P is not symmetric. One defines $S = P \dot{\cup} \breve{P}$ Def.
3) P is not reflexive. One defines $S = P \dot{\cup} I \upharpoonright C'P$ Def.
4) There are two basic relations, P and Q.
   a) $P \subset Q$ holds. Moreover, P is symmetric, transitive and reflexive, Q is symmetric and reflexive. One defines $S = P|Q|P$ Def.
   b) $P \subset Q$ holds, but P and Q do not have the properties required in a). One constructs from them two new relations, R and V, which have these properties. In particular, one introduces symmetry and transitivity by applying the transformations 2) and 3) and transitivity by applying the transformation: $R = P_*$ Def. One defines: $S = R|V|R$ Def.
   c) $P \subset Q$ does not hold. One defines: $S = P \dot{\cup} Q$ Def.
5) There are more than two basic relations.
   a) One of these relations, T, is implied by all the others: $U \subset T, W \subset T, Z \subset T, \ldots$ One defines: $P = U \dot{\cup} W \dot{\cup} Z, \ldots$ Def, which yields: $P \subset T$. Then, one applies (4a) or (4b) on these two relations.
   b) The condition in (a) is not satisfied. One divides the relations into two classes and takes both unions of these classes as new relations: $P = T \dot{\cup} U \dot{\cup} \ldots$ Def., $Q = W \dot{\cup} Z \dot{\cup} \ldots$ Def. If possible, the division takes place most conveniently when the constructed relations P and Q satisfy the condition in 4a) or, if this is not possible, the condition in 4b).

Let these transformation rules be here given in short without justification. That they lead to the desired result, i.e., that the new constructed relations and the relation *S* possess the required properties (symmetry, reflexivity, intransitivity) can be easily seen. According to the peculiarities of the case, other transformations, which lead to the same desirable result, turn out to be often more appropriate.

Now, <u>the problem of quasi-analysis</u> can be formulated as follows. <u>A set of elements is given and for every element the list of its similar elements. Find a description of this set that uses only these indications, but assigns to the elements quasi-components</u> (QCts) or quasi-characteristics <u>in such a way that every individual element can be described by itself</u>, <u>without reference to other elements,</u> according to its QCts. This problem can be solved by assigning to the elements QCts in such a way that two elements have a QCts in common if and only if they are similar. To minimize arbitrariness, it is required for QAl to state about the elements nothing more than what is already contained in the given lists. This way, two elements are represented as identical if and only if they are identical according to the given lists. Finally, following the principle of economy, it is required that no unnecessary QCts occurs in QAl. For this reason, in order to find a QAl of a given set of elements, a method must satisfy the following four basic requirements [axioms, *translator's note*].

<u>The four basic requirements.</u>

> (**I**) If two elements are similar, then they share at least one QCt.
>
> **(II)** If two elements are not similar, then they do not share any QCt.
>
> **(III)** If two elements *a* and *b* are "<u>similarity equivalent</u>" (i.e., *a* is similar to exactly the same elements that are similar to *b*), then they are "<u>QCt-equivalent</u>" (i.e., *a* and *b* possess exactly the same QCts).

The converses of (I), (II) and (III) do not need to be introduced as requirements. Rather, they follow from the above-mentioned basic requirements; cf. theorems (1), (2), (3) *infra*.

> **(IV)** There is no QCs whose removal leaves the requirements (I), (II) and (III) still satisfied.

It is shown through the method to be discussed in the sequel that these four requirements are <u>consistent one with the other</u> and are all satisfied. It can be seen that they are independent one of the other in analogy to the independence proof of the axioms of geometry (Hilbert).

> Every requirement is shown to be independent of the other by providing an example where it is not satisfied, but the others are. In the following example (I) is not satisfied, namely for *b* and *c*, while the other basic requirements are satisfied. *a, b, c, d* are elements and *aSb*, *bSc*, *cSd* hold (henceforth, we do not explicitly mention *aSa*, etc., and *bSa*, etc., since these similarities come respectively from the reflexivity and the symmetry of *S*). Also, α is a QCt of *a* and *b*, β is a QCt of *c* and *d*. In the next example, (II) is not satisfied, but the other requirements are satisfied. *a, b, c* are elements. *aSb*, *bSc* hold and α is a QCt of *a*, *b* and *c*. In the next example, only (III) is not satisfied, namely for any pair of elements. *a*, *b*, *c* are elements. *aSb*, *bSc*, *cSa* hold. Also, β and γ are the QCts of *a*, γ and α those of *b* and α and β those of *c*. In the next example, only (IV) is not satisfied, namely for β. *a, b, c, d* are elements. *aSb, bSc, cSa, cSd* hold. Also, α is QCt of *a*, *b* and *c*, β is QCt of *a* and *b*, γ of *c* and *d*.

The following seven theorems are consequences of the four basic requirements.

(1). Theorem. Let the four basic requirements be satisfied. If two elements share a common QCt, then they are similar.

This theorem is the converse of (I) and follows form (II).

(2). Theorem. Let the four basic requirements be satisfied. If two elements have no QCt in common, then they are not similar.

This theorem is the converse of (II) and follows from (I).

(3) Theorem. Let the four basic requirements be satisfied. If two elements are QCt-equivalent or similarity equivalent, then they are similar.

The second and the first parts of the theorem come from reflexivity of *S* and Theorem (1), respectively.

(4). Theorem. Let the four basic requirements be satisfied. If two elements are QCt-equivalent, then they are also similarity equivalent.

This theorem is the converse of (III).

> **Proof**. Let us assume that *a* and *b* are QCt-equivalent. By (I), for every element *c* which is similar to *a*, *a* and *c* share at least a certain QCt. Further, it follows from the QCs-equivalence of *a* and *b* that *b* and *c* share the same QCt. Hence, by (1), *bSc*. Therefore, every element which is similar to *a* is also similar to *b*.

(5). Theorem. Let the four basic requirements be satisfied. If two elements are not similarity equivalent, then they are not QCt-equivalent.

It follows from (4).

(6). Theorem. Let the four basic requirements be satisfied. If two elements are not QCt-equivalent, then they are not similarity equivalent.

It follows from (III).

(7). Theorem. Let the four basic requirements be satisfied. There is no QCt which is a "companion" of another QCt (a Ct or a QCt α is a companion of the Ct, or the QCt, β when α belongs only to elements to which β also belongs).

It comes from (IV), since such a QCt can be removed without violating (I) and (II).

2. *The first part of QAl: the similarity circles*

At first sight, the problem of QAl seems easy to solve: the relation of sharing a QCt between two elements easily takes the place of the given similarity relation. The difficulty lies in the fact that the similarity relation is not transitive while the relation of sharing a QCt is. Therefore, the attempt to finding a QAl by assigning a QCt α to an element *a* and the same QCt α to the elements *b, c, d* which are similar to *a* is clearly unsuccessful. It would indeed violate the requirement (II), since, from *aSb* and *aSc*, one cannot conclude *bSc*.

The subsequent discussion is intuitively based on a concrete **example**. The example is taken from the domain of the phenomenology of sense impression. During other researches, the investigation of this domain precisely suggested the development of the quasi-analytical method.

> **Example**. Let a set of 12 sounds, namely chords and individual tones, be given. Let us label the sounds, the elements of the set, *h, i, k, … t*: *h*= tone *d*, *i* =chord *d-f-a*, *k= c-e-g*, *l= c-e*, *m= f-a*, *n= d-f*, *o= c-e-a*, *p= c-f*, *q= c*, *r= d-a*, *s= g*, *t= c-g*. Let this composition be unknown. The 12 elements must be considered as indivisible and analysable only through quasi-analysis. For this, according to what we have said above, only the list of the pairs of elements which stand in the symmetric, reflexive, intransitive relation *S* are required. As such S-pairs let us introduce: *hi*, *hn*, *hr*, *im*, *in*, *io*, *ip*, *ir*, *kl*, *ko*, *kp*, *kq*, *ks*, *kt*, *lo*, *lp*, *lq*, *lt*, *mn*, *mo*, *mp*, *mr*, *np*, *nr*, *op*, *oq*, *or*, *ot*, *pq*, *pt*, *qt*, *st*. The opposite pairs *ih, nh, … ts,* do not need to be mentioned here, because of the symmetry of *S* (thus, we don't distinguish them from the former in the sequel). Similarly, we don't explicitly enumerate (reflexivity of *S*!) the identity *S*-pairs: *hh, ii, .. tt.* As one sees, *S* corresponds in the example to the familiar kinship of sounds, which is ordinarily called "agreement in (at least) a constituent tone" or "(at least) partial identity". This is emphasised here, however, only for the purpose of illustration, quasi-analysis does not pay attention to it. The elements must be indivisible in quasi-analysis which is based only upon the *S*-pairs. By using the example it can be shown that not only its results, but also many single steps of QAl are analogous to those of Al. This is another justification for choosing this name.

We call $\vec{S}'x$ the <u>similarity neighbourhood</u>[3] of x, i.e., the class of elements that are similar to x.

In our **example** we have: $\vec{S}'$h = [h, i, n, r], $\vec{S}'$i = [h, i, m, n, o, p, r], $\vec{S}'$k = [k, l, o, p, q, s, t], $\vec{S}'$l = [k, l, o, p, q, t], $\vec{S}'$m = [i, m, n, o, p, r], $\vec{S}'$n= [h, i, m, n, p, r], $\vec{S}'$o = [k, l, m, o, p, q, r, t], $\vec{S}'$p = [i, k, l, m, n, o, q, t], $\vec{S}'$q = [k, l, o, p, q, t], $\vec{S}'$r = [h, i, m, n, o], $\vec{S}'$s = [k, s, t], $\vec{S}'$t = [k, l, o, p, q, s, t].

(The following sections (8)-(12) and (16) are not required to apply the QAl method. Rather, they serve only to compare QAl with Al. Those who are interested only in the method and not in its justification can also omit them).

(8). Definition. $E_1 = \hat{x}\ \hat{y}\ (S'x \subset S'y)$ Def.

Thus, xE_1y means: the similarity neighbourhood of x is contained in the similarity neighbourhood of y.

Hence, we have in our **example**, $\vdash lE_1o$ but not $\vdash oE_1l$. Moreover, $\vdash sE_1t, \vdash qE_1t, \vdash hE_1i, \vdash mE_1i$.

Thus, a not symmetric relation E_1 is obtained from the symmetric relation S. This represents, in QAl, the following relation of Al: "y contains all the Cts of x".

(9). Definition. $E_2 = E_1 \dot{\cap} \breve{E_1}$ Def.

Thus, xE_2y means: E_1 and its converse hold between x and y. E_2 is the already mentioned relation of similarity equivalence. Indeed, we have:

(10). Theorem. $\vdash :. (x, y) : xE_2y . \equiv . \vec{S}'x = \vec{S}'y$

According to the basic requirement (III), there arises the problem of finding a QAl where any two elements which stand in the relation E_2 are QCt-equivalent. To compare it again with the Al, E_2 correspond to the Ct-equivalence of Al in many cases. This, however, does not always occur, namely, not if a Ct occurs in Al only as companion, which, by (7), is excluded in QAl.

In the **example** $\vdash \vec{S}'l = \vec{S}'q$ hold. Thus, $\vdash lE_1q$ and $\vdash qE_1l$, whence $\vdash lE_2q$. Hence l and q must be QCt-equivalent. But they are not Ct-equivalent, since the tone e belongs, as Ct, to l, but not to q. We shall later on see that this disagreement is dependent upon (7), and thence upon the basic requirement (IV).

(11). Definition. $E = E_1 \dot{\cap} - \breve{E_1}$ Def.

We have that E if E_1, but its converse does not hold. The relation E of QAl represents the following relation of Al: "x is Ct-equivalent to a proper part of y".

[3] For this translation choice see fn 2.

In the **example**, $\vdash l\ E\ o$, since $\vdash lE_1 o . o - E_1 l$. In Al, the Cts of *l* (the tones *c, e*) are proper parts of the Cts of *o* (*c, e, a*).

(12). Definition. $E_3 = \hat{x}\,\hat{y}\,(xE_1 y\,.\,\vec{E}'x = \Lambda)$ Def.

Thus, xE_3y means: the similarity neighbourhood of *x* is contained in the similarity neighbourhood of *y* and there is no element which stands in the relation *E* to *x*. The relation E_3 of QAl represents the following relation of Al: "to *x* belongs only one Ct and the same belongs to *x*" or "*x* is Ct-equivalent to an individual Ct of *y*" or, in short, "*x* is an individual Ct of *y*".

In the **example** we had $\vdash sE_1t$ and $\vdash qE_1t$. However, there is no element *u* in the given set such that *u*E*s* or *u*E*q*. Hence, $\vdash qE_3t$ and $\vdash s\ E_3t$ hold. For comparison, let us consider again Al. We have: *q* (namely the tone *c*) and *s* (tone *g*) are individual Cts of *t*.

One may conjecture that one could carry out QAl by assigning to every element *x* (as it were the "totality of its individual QCts") the class of the elements which stands in the relation E_3 to it, i.e., the class $\overrightarrow{E_3}'x$. However, as an example can clearly show, such a method would result in a violation of the basic requirement (I).

In the **example** one had: $\vdash hE_1i$ and $\vdash mE_1i$. There is no element *u* of the set for which *uEh* or *uEm* hold. Thus, $\vdash hE_3i$ and $\vdash mE_3i$. There is no other element like *h* and *m* which stands in the relation E_3 to *i*: $\overrightarrow{E_3}'i = [h, m]$. On the other hand, neither *h* nor *m* belong to $\overrightarrow{E_3}'p$. Indeed, $\vdash h - S\ p$ and $\vdash m - E_1p$, thus $\vdash m - E_3p$. Therefore, if one take $\overrightarrow{E_3}'x$ as the class of the QCses of *x*, then the elements *i* and *p* would share no QCts. But since $\vdash iSp$ the basic requirement (I) would not be satisfied.

If we compare with the Al, then we clarify the reason for the shortcoming of a QAl based on E_3: not every Ct occurs somewhere in isolation, i.e., as the only Ct of an element.

The attempted QAl of *i* into the individual QCts *h* and *m* would correspond to an analysis of the chord *d-f-a* into the two Cts *d* and *f-a*. Indeed, *f-a* would not be further analysable here since the tones *f* and *a* never occur in isolation. Now, as we will see, in this case the QAl precisely represents the tripartition of Al.

We define the "class of <u>similarity circles</u> (SC)":

(13). Definition. $sim = \hat{\beta}\,(x, y \in \beta \supset xSy :. (z) : (u). u \in \beta \supset uSz . \supset . z \in \beta)$ Def.

In words: we say β is a SC if the following conditions are satisfied: 1) any two members of β are similar; 2) if some *z* is similar to every member *u* of β, then z belongs to β. A SC is thus a complete set of elements that are similar one to the other.

To a SC corresponds, in Al, a class that contains all and only those elements which share a given Ct. However, not all such classes of Al have a corresponding SC in QAl, namely, not if the class is based on a Ct which is a companion of another. One cannot provide any analogue to such a class if the basic requirement (IV) must not be violated.

> In the **example**, the class [*k,l,o*] of elements, which share the tone *e* as Ct, is not a SC. For the latter condition of the definition is not satisfied: *q* does not belong to the set, but is similar to all its members. Since, later on, we draw the QCts out of the SCs, there is no QCt corresponding to the tone *e*. The reason for the above-considered disagreement between QAl and Al in relation to the tone *e* lies in the fact that *e* occurs only in elements in which the tone *c* also occurs. Therefore, *e* is a companion of *c*. Thence, according to the basic requirement (IV) and theorem (7), there is no QCt corresponding to it. On the other hand, $\delta = [h, i, n, r]$ is a SC. Its members are the elements that share the tone *d* as common Ct. Likewise, $\alpha = [i, m, o, r]$ is a SC, where the common Ct is the tone *a*. Similarly, $\gamma = [k, l, o, p, q, t]$ is a SC with the common tone *c*. One finds as remaining SCs: $\varphi = [i, m, n, p]$, $\xi = [k, s, t]$, $\pi = [i, m, o, p]$, $\rho = [i, m, n, r]$. Thus, $\vdash sim = [\delta, \alpha, \gamma, \varphi, \xi, \pi, \rho]$.

(14). Theorem. $\vdash :. (x, y): xSy. \supset. (\exists\alpha). \alpha \epsilon\ sim. x, y \epsilon \alpha$

In words: for every pair of similar elements there is (at least) a SC which contains them both.

> **Proof**. If there is no element *u* which is similar to *x* and *y* and different from both, then for $\alpha = [x, y]$ the theorem is satisfied. However, if there is such a *u*, then one constructs the class [*x,y,u*]. If there is no element *v* which is similar to every element of the class, then for [*x,y,u*] the theorem is satisfied. Otherwise, let us construct the class [*x,y,u,v*]. Continuing in this way, if the class of elements is finite, we must arrive, in a finite number of steps, at a class for which the theorem is satisfied.

(15). Theorem. $\vdash :: (x. y. \alpha):. \vec{S}'x = \vec{S}'y.\ \alpha \epsilon\ sim. \supset: x \in \alpha. \equiv. y \in \alpha$

In words: if two elements are similarity equivalent, then one belongs to the same SCs of the other.

> **Proof**. $\alpha \in sim$ and $x \in \alpha$ yield: any member of α is similar to x. Thus, since, $\vec{S}'x = \vec{S}'y$, it is also similar to y. Therefore, by (13), $y \epsilon \alpha$.

(16). Definition. Repr = $\hat{x}\hat{\beta}(\beta \in sim :. (y): y \in \beta. \supset. xE_3y)$ Def.

X Repr β means: *x* is a member of β and stands in the relation E_3 to every other member of β. In this case, we call the element *x* a "<u>representative</u>" of the

SC β. The relation *Repr* of QAl corresponds in Al to the relation: "β contains all and only those members which share the only Ct of *x*". As in the above-indicated cases, exceptions to this correspondence arise when the Ct in question is a companion of another. There are also SCs without representatives. In Al this means: the common Ct of the set does not occur in isolation. Since $\mathsf{D}'\,Repr$ indicates the class of those SCs which have at least a representative, $sim - \mathsf{D}'Repr$ indicates therefore the class of the SCs without representatives. Also, there are SCs with many representatives. These representatives in Al are either Ct-equivalent or they differ from one another through Cts that are companions of other Cts.

In the **example**, for the SC δ = [*h, i, n, r*] we have $\vdash h\ Repr\ \delta$. Indeed, $\vdash \delta \in sim. h \in \delta. hE_3 i. hE_3 n. hE_3 r$. In the Al: the Cs of *h*, namely the tone *d*, is what the members of δ have in common. Hence the expression "representative of δ" for *h*. A SC without representatives is, for example, α = [*i,m,o,r*], since none of its members stands in the relation E_3 to the other three. The reason for this is clarified by Al. The Cs common to all the members of α, namely the tone *a*, never occur in isolation. A SC with many representative is γ = [*k,l,o,p,q,t*]. Indeed, $\vdash l\ Repr\ \gamma. q\ Repr\ \gamma$ holds. In the Al, *l* and *q* differ from one another through the tone *e* as Ct, which is the companion of *c*.

as in the const. th.

Here again we could fall into a method that, at first sight, seems to be suitable for a QAl but does not lead to the desired result. Couldn't we assign to every element *x*, as QCts, the SCs to which it belongs? One clearly deduces from (13) that the essential basic requirements, namely (I), (II), (III) would be here satisfied. However, the attainment of the requirement of economy, (IV), would not be guaranteed. This can be very easily shown through the example. As we shall see, by avoiding this mistake one brings QAl and Al into a closer analogy.

In the **example**, according to the above-considered method, one would assign to the individual elements the following classes of SCs as their (attempted) classes of QCts: the class [δ] (i.e., the class whose only member is the SC δ) to the element *h*, the class [$\delta,\varphi,\alpha,\pi,\rho$] to the element *i*, [$\gamma,\xi$] to *k*, *l*: [$\gamma$], *m*: [$\varphi,\alpha,\pi,\rho$], *n*: [$\delta,\varphi,\rho$], *o*: [$\gamma,\alpha,\pi$], *p*: [$\gamma,\varphi,\pi$], *q*: [$\gamma$], *r*: [$\delta,\alpha,\rho$], *s*: [$\xi$], *t*: [$\gamma,\xi$]. The basic requirements (I), (II), (III) are here satisfied. But (IV) is not satisfied. For, even though we delete from the just given formulation the QCts π or ρ, or even all two, (I), (II), (III) still remain satisfied. These two SCs are also those without any analogue in the Al. While to the SCs δ, α, γ, φ, ξ correspond the classes of the elements sharing the tones *d, a, c, f, g*, respectively, neither the members of π nor those of ρ share a common Ct.

3. *The second part of QAl: the QCs-class as a result*

(17). Definition. $\mathrm{N} = \hat{\alpha}\hat{\beta}\ (\alpha \in \beta . \beta \subset sim :: (\exists x,y) : x \neq y . x,y \in \alpha :. (\gamma) : \gamma \in \beta . x,y \in \gamma . \supset . \gamma = \alpha)$ Def.

Thus, $\alpha N \beta$ means: β is a class of SCs; the SC α belongs to β; there are two different elements in α such that it is not the case that both of them belong to another SC in β. In this case we say: "α is relatively necessary with respect to β". From this follows: $\vec{N}'\beta$ is the class of the relatively necessary SCs with respect to β. If β contains all the SCs *(β=sim)*, then $\vec{N}'sim$ is the class of the "absolutely necessary SCs". $\alpha N sim$ means: α is an absolutely necessary SC.

In the **example**, let us now take $\beta = [\gamma, \phi, \alpha, \pi]$. Thus, $\vdash \gamma N \beta$ holds, since the *S*-pair *kt* and others occur only in γ, but not in ϕ, α, π. Similarly, $\vdash \phi N \beta$ because of the pair *mn*; $\vdash \alpha N \beta$ because of the pair *ir*. However, $\vdash \pi - N\beta$. Therefore $\vdash \vec{N}'\beta = [\gamma, \phi, \alpha]$. Moreover, $\vdash \vec{N}'sim = [\gamma, \delta, \phi, \alpha, \xi]$ holds, i.e., these five SCs are absolutely necessary SCs, in particular γ because of the pair *kl*; δ because of *hi*; φ because of *np*; α because of *or*; ξ because of *ks*. On the other hand, π and ρ are not absolutely necessary SCs, for none of them contains a *S*-pair which does not belong also to other SCs.

(18). Theorem. $\vdash\ N \in \varepsilon$

It follows from (17). In words: whatever stands in the relation *N* to a class belongs to it as member.

(19). Theorem. $\vdash :. (\lambda, \kappa, \alpha) : \alpha N \lambda . \kappa \subset \lambda . \supset . \alpha N \kappa$

In words: a SC is relatively necessary with respect to a class of SCs whenever it is relatively necessary with respect to another class of SCs where the former is contained (as subclass).

Let us now take for **example** $\lambda = [\gamma, \phi, \alpha]$ and $\beta = [\gamma, \phi, \alpha, \pi]$, as before. Then, $\vdash . \gamma N \beta . \phi N \beta . \alpha N \beta$ yields $\vdash . \gamma N \lambda . \phi N \lambda . \alpha N \lambda$

(20). Theorem. $\vdash . \vec{N} \mid \vec{N} = \vec{N}$

In words: the class μ of the relatively necessary SCs with respect to another class is identical with the class of the SCs that are relatively necessary with respect to μ itself.

(21). Theorem. $\vdash . \vec{N}'(\vec{N}'sim) = \vec{N}'sim$

It follows from (20). In words: the class $\vec{N}'sim$ of the absolutely necessary SCs is identical with the class of the SCs that are relatively necessary with respect to $\vec{N}'sim$ itself.

(22). Definition. suf $= \hat{\beta}(\beta \subset sim :. (x,y) : x \neq y . xSy . \supset . (\exists \alpha) . \alpha \in \beta . x,y \in \alpha)$ Def.

In words: β is said to be a "sufficient class of SCs" if for any two similar elements there is a SC α in β that contains them both.

(23). Theorem. $\vdash:.(\alpha,\beta):\alpha\in suf.\alpha\subset\beta.\supset.\beta\in suf$

In words: a class is a sufficient class of SCs whenever it contains a sufficient class of SCs as its subclass.

In the **example**, let us take $\kappa=[\gamma,\delta,\phi,\alpha,\xi]$. Hence, $\vdash\kappa\in suf$. Indeed, for any two similar elements there is at least one of these five SCs that contains them both. Let us take $\kappa_1=[\gamma,\delta,\phi,\alpha,\xi,\pi]$, $\kappa_2=[\gamma,\delta,\phi,\alpha,\xi,\rho]$, $\kappa_3=[\gamma,\delta,\phi,\alpha,\xi,\pi,\rho]$ (=sim). By (23) we have $\vdash\kappa_1,\kappa_2,\kappa_3\in hinr$. In our example there is no other sufficient class of SCs: $\vdash suf\ =[\kappa,\kappa_1,\kappa_2,\kappa_3]$.

(24). Theorem. $\vdash:.(x,y,\beta):xSy.\beta\in suf\ .\supset.(\exists\alpha).\alpha\in\beta.x,y\in\alpha$

It follows from (22). In words: for any two similar elements there is (at least) a SC in every sufficient class of SCs that contains them both.

(25). Theorem. $\vdash\ \vec{N}'sim\subset p'suf$

Follows from (17) and (22). In words: the absolutely necessary SCs belong to the intersection of sufficient classes of SCs and therefore to each sufficient class of SCs.

In the **example** we have $\vdash p'suf\ =\ \kappa$, since $\vdash.\kappa\subset\kappa_1.\kappa\subset\kappa_2.k\subset\kappa_3$. Further, $\vdash\ \vec{N}'sim=\ \kappa$ implies $\vdash\vec{N}'sim\subset\kappa$.

(26). Definition. $nec=\ \hat{\beta}\big(\beta\subset\vec{N}'\beta\big)$ Def.

In words: β is said to be a "<u>relatively necessary class of SCs</u>" ($\beta\in nec$) if it contains only those classes that are relatively necessary with respect to β itself.

In the **example** we had $\vdash.\gamma N\lambda.\phi N\lambda.\alpha N\lambda$, for $\lambda=[\gamma,\phi,\alpha]$. Therefore $\vdash\lambda\in nec$ holds, i.e., λ is a relatively necessary class of SCs. Let us take $\lambda_1=[\alpha,\xi,\pi,\rho]$, so we have again $\vdash\lambda_1\in nec$. Indeed, $\vdash\alpha N\lambda_1$ because of the pair *or*, which only occurs in α, but not in ξ, π or ρ. Similarly, $\vdash\xi N\lambda_1$because of the pair *st*; $\vdash\pi N\lambda_1$ because of *ip*; $\vdash\rho N\lambda_1$ because of *in*. For the above indicated classes $\kappa,\kappa_1,\kappa_2,\kappa_3$, $\vdash\kappa\in nec$ holds. Indeed, we have $\vdash\gamma N\kappa$, since the S-pair *kl* occurs only in γ, but not in δ, φ, α, ξ. Moreover, $\vdash\delta N\kappa$ because of the *S*-pair *hn*; $\vdash\phi N\kappa$ because of *np*; $\vdash\alpha N\kappa$ because of *or*; $\vdash\xi N\kappa$ because of *st*. On the other hand, $\vdash\kappa_1\sim\in nec$, for $\pi-N\kappa_1$, i.e., there is no S-pair in π which does not occur also in (at least) one of the remaining classes γ, δ, φ, α, ξ. Similarly, $\vdash\kappa_2\sim\in nec$, for $\rho-N\kappa_2$. In the same way, $\vdash\kappa_3\sim\in nec$.

(27). Theorem. $\vdash:.(x):x\in nec\ .\supset.\vec{N}'x=x$

Follows from (26) and (28). In words: a relatively necessary class of SCs is identical with the class of the relatively necessary classes with respect to itself.

(28) Theorem. $\vdash\ \vec{N}'sim\in nec$

It follows from (21). In words: the class of the absolutely necessary SCs is a relatively necessary class of SCs.

(**29**)**. Definition**. *sn* = *suf* ∩ *nec* Def.

If $\beta \in sn$, then β is said to be a "QAl-class". Thus, *sn* is the class of the QAl-classes. Such a QAl-class contains exactly sufficiently many SCs, to apply a QAl satisfying the four basic requirements, and no unnecessary SC.

(30). Theorem. $\vdash .\ sn \subset suf$

In words: every QAl-class is a sufficient class of SCs.

(31). Theorem. $\vdash .\ sn \subset nec$

In words: every QAl-class is a relatively necessary class of SCs. Both theorems follow from (29).

(32). Theorem. $\vdash . sn \subset \mathrm{Cl}'sim$

Follows from (29) and (22). In words: every QAl-class is a class of SCs.

(33). Theorem. $\vdash :. (x, y, \beta) : xSy . \beta \in sn . \supset . (\exists \alpha) . \alpha \in \beta . x, y \in \alpha$

Follows from (30) and (24). In words: for any two similar elements there is (at least) a SC in every QAl-class that contains them both.

(34). Theorem. $\vdash . \vec{N}'sim \ \subset p'sn$

It follows from (25). In words: the absolutely necessary SCs belong to every QAl-class. In general, *sn* has many members, i.e., many QAl-classes, so that there are many possible QAl of a given field of relations. An example of it will be discussed in the sequel (with the rules for a practical application of QAl). The following three propositions are concerned with the case where only one QAl is possible.

(35). Theorem. $\vdash : \vec{N}'sim \in suf . \supset . \vec{N}'sim \in sn$

Follows from (28).

(36). Theorem. $\vdash :\ \vec{N}'sim \in suf . \supset . sn \in 1$

By (34), every member of *sn* contains $\vec{N}'sim$. Now if *sn* contains more than one member, then the member of *sn* which is different from $\vec{N}'sim$ (for example ξ) must contain both $\vec{N}'sim$ and at least a SC which is not contained in $\vec{N}'sim$. Since, by assumption, any two similar elements already occur in at least one member of $\vec{N}'sim$, ξ would not belong to *nec* and therefore, by (31), not to *sn* either.

(37). Theorem. $\vdash :\ \vec{N}'sim \in suf . \supset . \vec{N}'sim = (\imath x)(x \in sn)$

Follows from (35) and (36). The content of the propositions (35), (36), (37) is: if the class of the absolutely necessary SCs is a sufficient class of SCs, then 1) it is itself a QAl-class, so 2) there is only one QAl-class and hence, in this case, 3) this class is the only QAl-class.

The premise of this proposition is fulfilled in our **example**. Indeed, we had: $\vdash \vec{N}'sim = [\gamma, \delta, \phi, \alpha, \xi]$ and for $\kappa = [\gamma, \delta, \phi, \alpha, \xi] \vdash \kappa \in suf$.Therefore, κ is the only QAl-class. This can be concluded from the previous results also: $\vdash . suf = [\kappa, \kappa_1, \kappa_2, \kappa_3]$, $\vdash \kappa \in nec$, $\vdash \kappa_1, \kappa_2, \kappa_3 \ \sim\in nec$ yield, by (29), $\vdash . sn = [\kappa]$.

If there are many QAl-classes, then we must decide which of them forms the basis of the QAl. If one of them has a smaller number of element than the others, then we will choose it. Otherwise, the choice is of no consequence, if it is not determined for reasons outside QAl, which are here not under discussion.

The QAl on the basis of a QAl-class assigns to every element the members of this class (i.e., SCs) as its QCts. Hence, the class of the QCts assigned to an element is called its "QCt-class". These QCt-classes for the individual elements are the result of the quasi-analysis. Actually, one can speak of the QCt-class of an element only as far as a definite QAl-class is concerned, i.e., a definite member of *sn*, since *sn* generally has multiple members. Let <β> indicate the relation of a QCt-class to its element with respect to the QAl-class β, then we must define:

(38). Definition. $< \beta > = \hat{x}\, \hat{y}(\beta \in sn . x \subset \beta . y \in p'x$ Def

Thus, D'<β> indicates the class of the QCs-classes with respect to the QAl-class β.

(39). Theorem. $\vdash : (\beta, x) . < \beta >' x \subset \beta$

It follows from (38). In words.: the QCt-class of an element with respect to a QAl-class β is contained in β.

In the **example** we had κ as the only QAl-class. Therefore, we assign the SCs (namely, the members of κ) to their respective members as QCts: accordingly, the QCt γ to the elements *k, l, o, p, q*, t; the QCt δ to the elements *h, i, n, r*; φ to the elements *i, m, n, p*; α to the elements *i, m, o, r*; ξ to the elements *k, s, t*. Thus, the following QCt-classes (with respect to κ) are assigned to the following individual elements: <κ>'*h* = $[\delta]$, i.e., δ is the only QCt of *h*; *h* is a representative of the QCt δ; <κ>'*i* = $[\delta, \varphi, \alpha]$, we say: the element *i* "consists", or "is composed of", the QCts δ, φ, α. Moreover, <κ>'*k* = $[\gamma, \xi]$; <κ>'*l* = $[\gamma]$; <κ>'*m*=$[\varphi, \alpha]$; <κ>'*n*= $[\delta, \varphi]$; <κ>'*o*= $[\gamma, \alpha]$; <κ>'*p*=$[\gamma, \varphi]$; <κ>'*q*=$[\gamma]$ <κ>'*r*=$[\delta, \alpha]$; <κ>'*s*=$[\xi]$; <κ>'*t*=$[\gamma, \xi]$.

4. *Testing the procedure against the four basic requirements*

(40). Theorem. $\vdash :. (x, \alpha, \beta) : x \in \alpha . \alpha \in \beta . \beta \in sn . \equiv . \alpha \in < \beta >' x$

Follows from (38). In words: if an element belongs to a member of a QAl-class, then this member belongs to the QCt-class of the element with respect to the same QAl-class; and vice versa. This gives rise to the following propositions.

(40a). Theorem. $\vdash:.(x,\alpha,\beta):\alpha \in <\beta>' x. \supset. x \in \alpha$

(40b). Theorem. $\vdash:.(x,\alpha,\beta):\alpha \in <\beta>' x. \supset. \alpha \in \beta$

(40c). Theorem. $\vdash:.(x,\alpha,\beta):\alpha \in <\beta>' x. \supset. \beta \in sn$

(40d). Theorem. $\vdash:.(x,\alpha,\beta):\alpha \in <\beta>' x. \supset. \alpha \in sim$
Follows from (40c), (40b), (32).

(41). Theorem. $\vdash:.(x,y,\beta):xSy.\beta \in sn. \supset. (\exists z). z \in <\beta>' x. z \in <\beta>' y$
Proof. Since $\beta \in sn$, we have $\beta \in suf$ by (29). It follows from (22) that for any two similar elements, in particular for the elements x and y of the premise, there is a SC α in β which contains them both. Thus, for $z = \alpha$, we have $z \in < \beta >' x. z \in <\beta>' y$, by (40).

(42). Theorem. $\vdash:.(x,y,\beta):(\exists z). z \in <\beta>' x. z \in <\beta>' y. \supset. xSy$
Proof. By assumption and (40a), we have $x \in z$ and $y \in z$. By (40d), we also have $z \in sim$. Thus, *xSy* follows from (13).

(43). Theorem. $\vdash:.(x,y,\beta):xSy.\beta \in sn. \equiv. (\exists z). z \in <\beta>' x. z \in <\beta>' y$
Proof. One of the two conditionals, out of which this equivalence is composed, corresponds to (41). In the other direction, i.e., the converse of (41), (42) yields *xSy* and (40c) yields $\beta \in sn$.

(44). Theorem. $\vdash:.(x,y,\beta):\vec{S}'x = \vec{S}'y.\beta \in sn. \supset. <\beta>' x = <\beta>' y$
(Indirect) **proof**. Suppose the proposition is false. Then, there is a member of the QCt-class (with respect to β) of an element, say *x*, which does not belong to the QCt-class of *y*. Let α be this member. Thus, we have $\alpha \in <\beta>' x. \alpha \sim \in < \beta >' y. \alpha \in \beta$. It follows from (40a) that $x \in \alpha$ and $\alpha \in sim$ follows from (40d). $\alpha \in \beta, \beta \in sn$ and (40) yield $y \sim \in \alpha$, which contradicts $\vec{S}'x = \vec{S}'y$, by (15).

(45). Theorem. $\vdash:.(x,y,\beta):<\beta>' x = <\beta>' y. \supset. \vec{S}'x = \vec{S}'y . \beta \in sn$
Proof. By assumption and (3), we have: $\beta \in sn$. Hence, by (41), $(u):uSx. \supset .(\exists v). v \in <\beta>' u. v \in <\beta>' x$. Now, by assumption again, $(z). z \in < \beta >' x. \supset. z \in <\beta>' y$. Therefore, we have $(u):uSx. \supset. (\exists v). v \in < \beta >' u. v \in <\beta>' x. v \in <\beta>' y$. Since, by applying (42), we obtain $(\exists v). v \in <\beta>' u. v \in <\beta>' y. \supset. uSy$, we have $(u). uSx \supset uSy$, thus $\vec{S}'x \subset \vec{S}'y$. Similarly, we prove $\vec{S}'y \subset \vec{S}'x$. Therefore, $\vec{S}'x = \vec{S}'y$.

(44) e (45) can be summarized in the following theorem:

(46). Theorem. $\vdash:.(x,y,\beta):\vec{S}'x = \vec{S}'y.\beta \in sn. \equiv. <\beta>' x = <\beta>' y$

(47). Theorem. $\vdash:: (\alpha,\beta):. \alpha \in \beta. \beta \in sn. \supset: (\gamma): (\exists x,y). xSy. x \neq y. x,y \in \gamma. \gamma \in \beta. \supset. \gamma = \alpha$

In words: if α belongs to the QAl-class β, then there are two similar elements which belong only to α and not to any other member of β different from α.

Proof. By assumption and (31), we obtain $\beta \in nec$. Thus, by (27), $\beta = \vec{N}'\beta$, whence $\alpha \in \vec{N}'\beta$ and therefore $\alpha N \beta$. By applying (17), the theorem holds.

The theorems derived above lead to the result that the basic requirements (I)-(IV) are satisfied. By (41), (I) is satisfied.

If, following (39), we examine the above indicated QCt-classes of the elements of our **example**, then we can confirm that they share at least one QCt for the S-pairs mentioned at the beginning of the example. Indeed, we have: the QCt-classes of *c* and *h* share δ, those of *h* and *r* share δ, those of *h* and *r* share δ, those of *l* and *m* α and φ, and so forth.

That the basic requirement (II) is satisfied follows from (42), using the contrapositive of its conditional (i.e., exchanging and negating both of its parts). In summary, (43) means that (I) and (II) are satisfied.

The examination of the example shows that no other pairs of elements than *S*-pairs share a QCt.

(44) means that the basic requirement (III) is satisfied. The converse of (III), which is also required, but does not need, as was shown above, to be indicated by itself, is satisfied by (45). That (III) and its converse are satisfied is summarized in (46).

In our **example**, only the pairs *kt* and *lq* are similarity equivalent. The same pairs are also QCt-equivalent: $< \kappa >' k =< \kappa >' t = [\gamma, \xi], < \kappa >' l =< \kappa >' q = [\gamma]$.

(47) means that for any QAl-class, and therefore any possible QAl, and any QCt α there is a *S*-pair whose elements have no other QCt in common than α. It follows that, whatever QCt one removes, there would be a *S*-pair whose elements would have no QCt to share. Hence, the basic requirement (I) would not be satisfied. Therefore, the basic requirement (IV) is satisfied.

In our **example**, in order to prove $\kappa \in nec$, following (26), we have shown that for any member of κ there is a *S*-pair occurring only in this and not in the other members. Therefore, whatever member of κ we remove, there would be two similar elements with no QCt to share anymore. This would violate (I). Therefore, in our example, the requirement (IV) is satisfied.

5. *Comparison of quasi-analysis with analysis*

The analogy between both methods has been repeatedly emphasised at each step. Using the example, it can be very easily shown that the result of QAl, in opposition to the simpler possibilities mentioned earlier, not only satisfies the four basic requirements, but it also achieves a more detailed correspondence to Al. It turns out that all the Cts have a QCt as their strict analogue, with the exception of the Cts that are companions of others and are not allowed, therefore, to have any analogue in a QAl that satisfies the four basic requirements.

> In our **example**, among the Cts of the Al, namely the tones *c, d, e, f, a,* the tone *e* is a companion of *c*. Thus, it has no corresponding QCt. With this latter exception, the QAl corresponds strictly to the Al. If, in the QCt-classes of the elements that the QAl has found, we replace the QCt γ by the tone *c*, similarly δ by *d*, φ by *f*, ξ by *g*, α by *a*, then we have $<\kappa>'\ h = [\delta]: [d]$, $<\kappa>'\ i = [\delta, \phi, \alpha]: [d, f, a], <\kappa>'\ k = [\gamma, \xi]: [c, g]$, and so forth. Accordingly, for every