The significance of *Quasizerlegung* for Carnap's *Aufbau* and scientific philosophy in general

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1. *Introduction*

In January 1923 Carnap completed a manuscript with the lengthy title *Die Quasizerlegung. Ein Verfahren zur Ordnung nichthomogener Mengen mit den Mitteln der Beziehungslehre* (henceforth *Quasizerlegung*).[1] *Quasizerlegung* can be considered, together with a few other manuscripts of the early 1920s, as important groundwork of Carnap's first opus magnum *Der logische Aufbau der Welt* (1928; Eng tr. 1967, henceforth *Aufbau*) (see Proust 1986, Eng.tr. 1989: section 4; Mormann 2009). The content and text of *Quasizerlegung* can be easily reckoned as a theoretical base for many passages of *Aufbau*. The treatment of quasi-analysis in the *Aufbau* and the distinction between property and relational descriptions (§§10, 71-74, *passim*) were, indeed, already examined in the previous typescript (1-2). Some examples of property and relational description (see for instance *Aufbau*: 20, 114-116) were also developed in detail there (1, 5). Similarly, the formalism that the author introduces in *Aufbau* (§97) and explains in the *Abriss der Logistik* (Carnap 1929) was already applied in the former work.

The history of the reception of *Quasizerlegung* is complicated. Although it has never been published "officially", it has attracted the attention of quite a few readers through the decades. Without claiming to be exhaustive, let us mention the following philosophically relevant episodes.

According to Carnap (1957), the contents of the manuscript were debated for the first time at the "Erlangen Conference" in March 1923 (cf. Del Sordo 2016: 205-6). As Damböck (2021: 23-7) emphasizes, the content of *Quasizerlegung*, together with those of other early works, conveys the core idea of what can be characterized as Carnap's *Herzensprojekt*, in accordance with a letter from Carnap to Franz Roh in 1925 (23). By this "*Herzensprojekt*",

[1] The unpublished manuscript is preserved at the Archives of Scientific Philosophy (ASP), Hillman Library, Carnap papers, University of Pittsburgh (RC-081-04-01).

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he meant the proficient application of contemporary theory of mathematical structures to epistemological and ontological problems arising from German philosophy across the nineteenth and twentieth centuries (see Mormann 2016; Schnädelbach 1983, Eng.tr. 1984; cf. Damböck 2021: 21, 37-39). As Carnap notated in his diary (see Carus 2007: 157-158), his discussion partners in Erlangen, among them Hans Reichenbach, Kurt Lewin and Paul Hertz, did not understood very much the intention of his project. This course of misunderstanding has probably driven the author to abandon his *Herzensprojekt* at least in its original shape (cf. the conjecture of Damböck 2021: 27).

More than a decade later, *Quasizerlegung* became a topic in the correspondence between Carnap and Goodman dated January 1938 (see Proust 1986, Eng.tr.: 191-193). After Goodman's famous criticism of quasi-analysis (see Goodman 1951: ch. 5), Joelle Proust unearthed *Quasizerlegung* in 1986 in her book *Questions of Form.* She reconsiders the piece in its relevance to the *Aufbau* project and argues that Goodman's 1951 criticism rests on an underlying misinterpretation of the young Carnap's philosophical perspective (Proust 1986: Eng.tr.: 191-193). In this manoeuvre, Proust casts a renewed productive sidelight on the *Aufbau's* formal method by revealing that the axiomatic apparatus of quasi-analysis is much richer in *Quasizerlegung* than in *Aufbau*. While introducing his formal method in *Aufbau* (§80) Carnap mentions indeed only two of the four axioms applied in the manuscript (3). Elaborating Proust's philosophical investigations, Mormann in 1994 showed that axioms of *Quasizerlegung* can be interpreted as axioms for a structural representation. Elaborating this result, quasi-analysis can be aligned therefore with some of the main algebraic results of the 20th century mathematics (see Mormann 2009: 277 *passim*; Davey *et al.* 2002: chs. 5, 11). By means of further historico-philosophical investigations Mormann (2016: especially 118-129) also traces the origin of Carnap's quasi-analysis back to the German cultural *milieu* of *Lebensphilosophie* and in particular to the philosophy of neutral monism that authors like Mach, Ziehen and Avenarius among others developed around 1900.

A minor obstacle for the contemporary reader's understanding of *Quasizerlegung* (and more generally of the more formal passages of the *Aufbau* and its significance) resides in the fact that Carnap used in these texts an outdated formalism of logic and the theory of relations that is essentially that of Russell/Whitehead's *Principia Mathematica.* Actually, Carnap used not more than the most elementary terminology of propositional logic and theory of relations that can easily be translated into the nowadays more familiar set-theoretical terminology. Moreover, the "theorems" of *Quasizerlegung* are almost always logically and mathematically rather trivial reformulations of the definitions and need no more than few lines to be proved.[2]

A short presentation of the symbolism employed in *Quasizerlegung* can be found in his booklet *Abriss der Logistik. Mit besonderer Berücksichtigung der Relationstheorie und ihrer Anwendungen* (Carnap 1929, henceforth *Abriss*).[3] In the *Aufbau* (§97) one can find a short list of the terminology used in this work, *Quasizerlegung* and other early writings of the author. In sum, the contemporary reader should have no unsurmountable difficulties to translate all formulas that Carnap used in these works in the more familiar terminology of informal set-theory. Be this as it may, in order to render *Quasizerlegung* more easily accessible for the contemporary reader we have added an appendix containing some (hopefully) useful hints and explanations that should make reading the manuscript more easily.

After these preliminary remarks let us now come to a crucial point, namely, a compelling argument why – after all – *Quasizerlegung* deserves to be carefully studied. In order to make plausible the claim that the manuscript has more to offer than an ingenuous formalism without philosophical significance, it is necessary to show that its perspective can help for a better understanding of significant concepts of contemporary philosophy and science. This is exactly what we want to sketch in the following. More precisely, we claim that the method of Carnapian quasi-analysis (as presented in its most elaborated form in *Quasizerlegung*) may be understood as a prototype of a promising mathematical philosophy in the sense that recently was explicated by Leitgeb (2012). Mathematical philosophy in this sense can be traced back to Russell's trail-blazing *Our Knowledge of the External World as a Field for Scientific Method in Philosophy* (Russel1 1914). We'd like to put forward the thesis that mathematical philosophy constitutes a current research field that closely inherits the original spirit of *Quasizerlegung* and Carnap's *Herzensprojekt* program in general. Indeed, mathematical philosophy promotes a view according to which philosophy is neither an ancillary discipline accompanying science, as it is in the analytic approach of logical empiricism, nor a part of science itself, as in the naturalized epistemology program of Quine and others (Leitgeb 2012: 267-268). Quite the contrary: mathematical philosophy ostensibly exhibits a close affinity with the idea of *Quasizerlegung* and Carnap's *Herzensprojekt* in general, by pursuing a philosophical research through mathematical, logical, and scientific methods and maintaining at the same

[2] In modern theory of relations there are non-trivial contentful theorems (see Maddux 1991; 2006; Givant 2017). Some interesting theorems concerning quasi-analysis and the complexity of similarity structures have been proved by Brockhaus 1963. They are discussed in Mormann 2009.

[3] Since some years a free electronic copy of *Abriss* in pdf is available in the internet.

time philosophy as a discipline in its own right, possessing its own problem, concepts and history (*ibid*.: 268-269).

The fundamental significance of *Quasizerlegung* for mathematical philosophy has been largely ignored so far. Either the early Carnap is conceived as a (proto-)analytical philosopher who achieved his philosophical maturity only in his later works, such as *Logical Syntax of Language* or *Empiricism, Semantics, and Ontology*. Or, this point of view is preferred by more historically inclined philosophical spirits, the early Carnap is conceived as a somewhat peculiar neo-Kantian philosopher. As we want to show both accounts fail to meet the full significance of *Quasizerlegung* for Carnap's philosophy in particular, and for modern scientifically-minded philosophy in general. Our proposal is, instead, to conceive the manuscript as the prototype of a scientifically-minded mathematical philosophy where two strands of thought come together, namely, the theory of representation, from working mathematics, and neutral monism, from philosophico-scientific research. The general target of our investigation is the theoretical significance of the recent historico-philosophical discoveries according to which quasi-analysis originates as both representation theorem (Mormann 2008) and means to reconcile the dichotomy between *Leben* and *Geist* sprouted from the soil of the early 20th century German thought in *Lebensphilosophie* and of neutral monism in particular (Mormann 2016). Indeed, given that quasi-analysis turns out to be a representation theorem, what does such a theorem serve for in neutral monism? How can it emerge over the course of a philosophy that, like neutral monism, elaborates ideas and tendencies that traditionally drift far apart from the analytic leanings of Carnap's later thought (cf. Schnädelbach 1983, Eng.tr.: ch. 5)? Now, formulating a definite and comprehensive answer to these philosophical questions is a profound task of inquiry and necessarily exceeds the scope of a single paper. Our particular aim is more modest consisting in paving the way for this kind of answers. In order to contribute to the reception of *Quasizerlegung* in this sense, the structure of this paper is as follows. In section 2, we specify some reasons for considering the philosophical relationship between neutral monism and representation theorems as a difficult, but highly rewarding issue of research. In section 3, we pretend to open the way for such an account by uncovering relevant points of convergence between the philosophical and mathematical enterprises of neutral monism and representations. In section 4 we conclude with some general remarks on the significance of *Quasizerlegung* for Carnap's philosophy and scientifically minded contemporary philosophy in general. In the appendix, we briefly explain the formalism that Carnap employed in the manuscript, in particular the formal concept of relation and the basic notions of the calculus of relations.

2. *Early Carnap's project: bringing together representation and neutral monism*

In order to argue for the relevance and possible fruitfulness of early Carnap's project of a mathematical philosophy (realized only in a preliminary and incomplete form in *Quasizerlegung* and other early manuscripts) let us comment upon some pieces of Carnap's earliest philosophical production of the 1920s. Admittedly, a lot of guesswork and speculation is involved in this endeavour. An essential ingredient of Carnap's project was geometry in a general sense. More precisely, geometry understood as synthetic geometry as a general theory of "*Ordnungsgefüge*"(cf. Mormann 2003: 47-50). Indeed, in *Quasizerlegung*, in *Der Raum*[4] (Carnap 1922b: ch. I, The Formal Space), and in the *Aufbau* (§70) Carnap used the very same example of color stripes and their similarity relations as an argument to argue for a geometry as a general (even universal) representational theory of order.

The concept of representation has, persistently, maintained a central position in the history of philosophy. Consequently, it has become a highly ambiguous concept with many different, even inconsistent meanings. In order to forestall any unnecessary misunderstandings, we want to point out from the outset that we subscribe to a <u>monistic</u> concept of representation that emphasizes the unity of the representational realm.

As the Neo-Kantian Ernst Cassirer (a frequent critical reference point of many logical empiricist such as Schlick, Frank, and Carnap in the first third of 20th century) has pointed out, many metaphysical doctrines tend to separate the domains of the representing and the represented, often conceived as the domain of "thought" ("*Denken*") and the domain of "things" ("*Dinge*") (cf. Cassirer 1910: 359). Thereby different "natures" are ascribed to both domains leading to the well-known riddle of how human knowledge is able to bridge the abyss between the allegedly totally separated two domains. In this sense, also in Cassirer's Neo-Kantianism epistemology an element of neutral monism can be identified.[5]

[4] In *Der Raum* (Carnap 1922b) Carnap explicitly mentioned the conceptual affinity of quasi-analysis with synthetic geometry: He pointed out that a classical theorem of Desargues may be understood as a quasi-analytical representation. A direct reference to geometrical representation theory is, however, neither mentioned in *Quasizerlegung* nor in the *Aufbau*. Thus, up to now, the philosophically crucial connection between geometry, order theory, quasi-analysis, and constitutional theory of the *Aufbau* has been rather ignored by Carnap scholars.

[5] The monist thesis of "representation first" may be backed also anthropologically. According to Ian Hacking: "The first peculiarly human invention is representation. Once there is a practice of representing, a second-order concept follows in train. This is the concept of reality, a concept which has content only when there are first-order representations. It will be protested that reality, or the

In order to better understand to what the primacy of representation amounts it is expedient to have a more detailed look on what a successful representational practice is expected to offer. We'd like to put forward the thesis that representations in mathematics and other cognitive enterprises aim at representation theorems. This is of critical importance since representation theorems can be characterized as monistic representations that assume the represented objects as primitive entities. The kind of representations that feature in mathematical representation theorems is, however, somehow unusual from both the analytical and ordinary points of view. Indeed, ordinary and analytical representations – one may think of representation as it arises, for example, in the philosophy of the first Wittgenstein – assume the represented object not to be primitive, but rather embedded in a domain of other objects that may serve in practice as representing props. As we shall see in a moment, representation theorems behave quite differently. They ferret out, in fact, representing props only via exploration of the represented object, since other domains of objects fall out of reach, by assuming primitivity.

What are representation theorems and what are they good for? A valuable attempt to answer these questions in an accessible, but informative and substantial way has been made by Davey and Priestley in their textbook on lattices and order (Davey *et al.* 2002). Even if they only deal explicitly with a special class of ordered structures, namely, lattices. In fact, their arguments apply to a much wider class of representations and representation theorems.

To be specific, a representation theorem for a class of lattices aims at a better theoretical and practical understanding of a class L of lattices. This is to be achieved by finding for the members of L sets P of basic <u>building blocks</u> ("atoms", "prime elements", "irreducible elements" etc.). These building blocks are either elements of the lattices or generated by them.

With respect to a lattice L, its "generating" set P has to fulfil (in some sense to be specified) the following requirements (*ibid.*: 112):

(A) The elements of P are readily identifiable. The cardinality of P should be as small as possible;
(B) The ordered set P should determine L in a unique way;
(C) The construction of L starting from P should be executable in a simple way.

The conditions (A) – (C) are to be interpreted as general guidelines or blueprints for conceptual constructions. They can be carried out and evaluated in various ways. For instance, with respect to (A) in many cases it is not uniquely

world, was there before any representation …. Of course. But conceptualizing it as reality is secondary" (Hacking 1983: 136; cf. also Rheinberger 2010: ch. 6).

determined what entities have to be chosen as appropriate "building blocks". Usually, it is not at all obvious what objects have to be chosen as elements of P. Only in rare cases it is obvious what the "atoms" are to be.

To be specific, let us consider some examples. For instance, in the case of an "atomic Boolean algebra" B it is rather clear that the elements of its generating set are to be taken as its atoms (smallest non-zero elements). However, already for non-finite Boolean algebras it is no longer possible to assume the existence of atoms in the usual sense. In general, for non-Boolean lattices atoms may not be available. Instead, appropriately chosen structures such as prime ideals have to be found that can play the role of building blocks. This task may require a high degree of ingenuity and technical skill. This is shown, in a particularly impressive manner, by Stone's famous Representation theorem that is to be considered as a paradigmatic case of a representation theorem *überhaupt*.

As Davey *et al*. also point out (2002), the requirements (A)-(C) have to be evaluated in a flexible way. For instance, in the case of Birkhoff's representation theorem, the requirement (A) of "smallness" is clearly satisfied (*ibid*.: 121). On the other hand, by taking Stone's representation theorem, "smallness" has to be evaluated with a grain of salt. In particular, when L is an infinite complete Boolean algebra, P is anything but small. Indeed, by the theorem of Balkar-Franěk (see Koppelberg 1989: 196), the cardinality of P is equal to the cardinality of L. Nevertheless, also in this case, the constitution of L from P is to be considered as an important conceptual achievement for other reasons.

Further problems arise in interpreting (A) as a guiding principle from the "easy identifiability" condition on P. Surely, one can think of this condition as satisfied by building block structures in Birkhoff theorem and finite Boolean algebra representation (Davey *et al*. 2002: 116-121). Its fulfilment becomes seriously debatable, however, whenever one has to appeal to the axiom of choice, or other maximality principles, to prove the existence of prime ideals or prime filters as is the case for Stone's representation theorem and many other modern theorems of this kind. This situation occurs in lattice theory, in the case of Stone and Priestley representation theorems (*ibid*.: chs. 10-11), as well as quasi-analysis (Mormann 2009: 259).

Let us now consider requirement (B). The theorems of Birkhoff and of finite Boolean algebras undoubtedly meet it (Davey *et al*. 2002: 114-116). To render (B) more precise, one should probably strengthen its criterion by also adding fundamental relations other than the order one. Let us finally consider (C). As Davey *et al*. (2002) emphasises, one cannot consider it as overall satisfied even in the realm of lattice theory. A general representation theorem of finite complete lattices would hardly meet it, indeed (*ibid*.: 168).

If one considers representation theorems from a wider philosophical point of view, then it might be convenient to interpret (C) in terms of an epistemic economy principle of some kind. In this case, since the fourth axiom of quasi-analysis too (3) is interpreted as an economy requirement (Mormann 2009: 253), the same axiom can be considered as an instance of (C). Accordingly, just as in lattice theory, (C) is not always satisfied by quasi-analysis either, as some additional conditions might conflict with it (*ibid.*: 262, *passim*).

Taking into account the difficulty of finding appropriate building blocks for many apparently simple mathematical representations, it might not be surprising that the analogous task of finding appropriate "neutral elements" in the realm of philosophy in general features a similar difficulty. Neutral monism is a comprehensive philosophy which includes both epistemological and ontological theses. According to it, the world we live in is entirely constituted by systems of "neutral" elements. To borrow a famous (or notorious) Neo-Kantian pun the neutral elements are not "*gegeben*" to us, rather, the task of finding them is "*aufgegeben*".

Neutral elements are identified by being structural, qualitative and pre-cognitive entities (see Del Sordo 2021: ch. 2). Among these features, that of "being pre-cognitive" is allegedly the most puzzling. Pre-cognitive nature renders, indeed, neutral elements elusive to any form of cognitive attitude purportedly focused on them. Because of this, the matching of our common believes to the theses of neutral monism is not an easy task to carry out, as Textor (2021: 33-37) recently showed. Such an epistemic difficulty has often led philosophers to weaken the strength of its theory. By virtue of the elusiveness of neutral elements, Tully (2003: 337-338), for instance, ends up reducing neutral monism to a metaphor or, at most, to a very abstract and formal hypothesis. Such interpretation does not provide, however, a coherent view of the movement. Indeed, it basically neglects the fact that its exponents were deeply engaged in finding appropriate theoretical strategy to overcome the epistemic elusiveness of their fundamental elements. Mach held, for example, that one day a future physiology would have empirically grasped what neutral elements essentially are (see Mach 1896, Eng. tr. 1898: 212; Banks 2003: 134). In this respect, Russell too seemed to share, at least in some passages, a line of thought allied with Mach's (see for example Russell 1927: 281-282).[6]

If we confine our discussion to the standard authors of neutral monism, i.e., Ernst Mach, William James and Bertrand Russell, then the syntheses of

[6] Insofar as its content is currently under debate, we must be cautious in making this statement. Actually, in the scholarship of neutral monism (see for example Wishon 2021: 139-141), Russell's neutral entities afford also a reading in terms of inscrutables, which renders them ungraspable by any form of knowledge.

the movement proposed by Banks 2014 and Stubenberg 2016 can be arguably considered as a unitary meta-theoretical account of the movement. Both have the drawback, however, not to put neutral monism in a wide enough historico-philosophical context. Fundamental elements with the same characteristics of the neutral ones, i.e., being qualitative, structural and subjectless, can be encountered indeed in the philosophical perspectives of Husserl's phenomenology and Bergson's metaphysics, to mention just two (see Schnädelbach 1983: Eng.tr., ch.5, also 148, *passim*). Husserl and Bergson, which one may eventually qualify as non-standard authors of neutral monism, were not content, just as Mach and probably Russell, with merely abstract or metaphorical proposals. They rather developed sophisticated methodologies, respectively based on *epoché* and intuition, to cognitively grasp the essence of their neutral fundamental entities (see for example Husserl 1913: § 63; Bergson 1911: ch. 3).[7] After all, an early version of quasi-analysis may be found in the work of Ziehen (1913: 3; cf. also Mormann 2016: 116) as a suitable philosophical method to address an epistemic challenge that was analogous to the one addressed by Mach, Husserl or Bergson. Quasi-analysis was meant, indeed, to free the constitution of reality in neutral monism from undue cognitive or epistemic assumptions (Ziehen 1913: 1-2, 177-178), a target that, according to Ziehen (1920: 217), phenomenological and scientific methods hinging upon extraordinary intuition or future physiology were not able to perform.

The considerations of this section entail that the theoretical meaning of the historico-philosophical origins of Carnap's quasi-analysis has to be explored by answering the questions: in what sense does quasi-analysis arise as an alternative philosophical method to those of other authors, and how can it eliminate undue epistemic assumptions within the constitution of reality in neutral monism? To answer these questions, plausible solutions to the above-mentioned meta-theoretical difficulties must be worked out. To this end, additional mathematical and philosophical topics should be deeply unfolded. Indeed, it is a matter of following the scientific development of representation theorems, perhaps using the formalism of category theory (for references on this see Davey *et al.* 2002), and the metaphysics and epistemology of order as it was developed around 1900 (for references and insights on this issue see Ziche 2016). Delving into these questions lies beyond the scope of this paper. Even so, we hope this section has exposed how, in spite of its difficulties, a meta-theoretical examination of neutral monism, representation theorems and their partnership may be a highly rewarding research topic.

[7] For a detailed study of the general connection of Husserl and Bergson with neutral monism their most relevant works are *Ideas 2*, *Analysis Concerning Passive and Active Synthesis* and *Matter and Memory.*

3. *Neutral monism and representationalism: towards a common program*

Some basic points of convergence between neutral monism and mathematical representation are undeniable. First of all, they both carry out a three-part theoretical program comprising *pars destruens* and *pars construens*. They start with complex relational entities, like the natural world or abstract algebras, whose constitution they want to clarify. To gain a better understanding of such complex entities, neutral monism and mathematical representation reduce them to building block structures (*pars destruens*). Forgetting[8] provisionally the information one has about the entities in their unreduced form, they finally elaborate perspicuous reconstructions of them by using only information provided by the building block structures (*pars construens*). This three-part program can be condensed to the following schema:

	Complex Structural Entity	Building Block Structures *(pars destruens)*	Perspicuous Reconstruction *(pars construens)*
Neutral monism	Natural World	Neutral Elements	Neutral Elements (Perspicuous) Natural World
Mathematical representation	Abstract Structures	Prime Elements	Prime Elements (Perspicuous) Abstract Structures

Within this schema, one can track down three additional patterns of affinity between neutral monism and mathematical representation.

1. The items, e.g. everyday natural objects or algebraic elements, which the complex entity consists of, turn often out to be systems of building block elements (whether neutral or prime) (cf. Banks 2014: ch. 1; Davey *et al.* 2002: chs. 5, 11).
2. In mathematical representation building blocks tends to be relation-

[8] The term "forgetting" has been chosen on purpose here. "Forgetful functors" are a basic concept of category theory that may be considered as a generalization of lattice theory playing a prominent role in the foundations of mathematics, informatics, and theoretical computer science (see for instance Simmons 2011: 76). "Forgetful functors" apply to those structures whose relations or operations must be set aside. Even for philosophers with only a rudimentary education in history of philosophy it is impossible not to detect an epistemological affinity between the operations of 'forgetful functors' and '*epoché*'. They both act, indeed, in such a way as to put some previously acquired knowledge into brackets.

ally, or structurally, poorer than the initial complex entities (see for example *ibid.*: 121, 262). Hence, we obtain that: if one epistemologically assumes that knowing is a matter of connecting, linking or ordering entities (see for example Ziche 2016: 91, *passim*), then the relational poorness of prime elements might simulate, or approximate, a pre-cognitive condition of sorts. If this is right, then one may also argue that the building block structure of quasi-analysis originally serves the purpose in neutral monism of emulating the elusive pre-cognitive condition of neutral elements. This hypothesis is, however, momentarily difficult to ascertain or generalize. From the historico-philosophical point of view, its ascertainment requires further examinations that can be unfolded better in papers *ad hoc*. Also, from the mathematical point of view, it cannot be generalized either, for the problem of determining the building block structures, i.e., problem of representability (see Davey *et al.* 2002: 261), still remains unsolved in many cases.

3. In the *pars construens*, both mathematical representation and neutral monism must comply with an economy requirement of some kind. Concerning mathematical representation, we have already seen in section 2 an economy requirement showing up both in (C) and in the fourth axiom of *Quasizerlegung* (3). Concerning neutral monism, such requirements have been applied at least in the form of Mach's normative economy of thought (Banks 2004: 24-5), where natural world is taken to be a parsimonious epistemic construct underpinned by an ontological array of pre-cognitive and chaotic neutral elements.

The above-considered three-column schema and points 1-3 do not contain a full-fledged exploration of the relationship between neutral monism and representation theorems. Nevertheless, they can be used as an entering wedge to further understand the philosophical origin of Carnap's quasi-analysis and envisage applications of mathematical representation in philosophical projects encompassing, like neutral monism, both ontology and epistemology.

4. *Concluding remarks*

Quasizerlegung is a piece of philosophy that defies straightforward classification. On the one hand, its formal aspects led some commentators to classify it as a sample of (early) analytic philosophy. If analytic philosophy is characterized, however, as the philosophical current according to which a philosophical account of thought can be attained only through language (see Dummett 1994: 4), then *Quasizerlegung ipso facto* does not appear to be very analytic. Indeed, the text features an amalgam of heavily loaden philosophical concepts, connected with metaphysical irrationality and ontological neutrality. On the other hand, it would be too simple to interpret it as a work of neo-Kantianism or Husserlian phenomenology. Mormann (2006: 27-33 ff.) and Damböck (2021: 39-41) have shown that interpretations of this kind, for instance Friedman (2000) and Mayer (2016), overlook many essential features of early Carnap's work.

In sum, interpretations that force young Carnap's work into ready-made categories either of analytic philosophy, neo-Kantianism, or phenomenology turn out to be Procrustean beds for this text. This nourishes the suspicion that classical philosophical categories are too rigid to capture *Quasizerlegung*'s true meaning. One may conjecture that other less known and subterranean traditions are at stake here – one may think for example of the so-called "lost" neo-Kantian tradition (Beiser 2014) of Herbart, whose influence on neutral monism only recently has been re-appreciated (see Banks 2003: ch. 3)

Until today, some authors, although engaged in Carnapian scholarship, simply ignore the concept of quasi-analysis. For instance, in Carus influential book, *Carnap and Twentieth-Century Thought. Explication as Enlightenment* (Carus 2007) the concept of quasi-analysis does not even appear once, although this book claims to deal with the significance of Carnap's philosophy in general. Chalmers too, who, in his bulky *Constructing the World* (2012), explicitly pretends to resuscitate Carnap's "Aufbau-program", but does not mention the method of quasi-analysis at all.

In other publications dealing with Carnap and Carnapian philosophy quasi-analysis scores better and pops up quite often. Nevertheless, its role is usually restricted to a sort of philosophical Cinderella. The concept is briefly mentioned, but almost never treated in detail. In most chapters of Damböck *et al.* (2021) anthology, *Der junge Carnap in historischem Kontext 1918 - 1935*, quasi-analysis either is not mentioned or, when it is mentioned, is introduced *ex abrupto* without any formal or informal explanation of what it is about or is not. This treatment of a philosophical Cinderella was, so to speak, familiar to *Quasizerlegung* from the very beginning. In fact, as we have already seen, Carnap himself abandons the content of the manuscript together with his *Herzensprojekt* after being misunderstood by colleagues in Erlangen and advised by Reichenbach to bring the philosophical focus away from the overly general attitude (according to Reichenbach) of his early *Konstitutionstheorie* (see Damböck 2021: 25-26).

In sum, the history of *Quasizerlegung* and, more generally, of quasi-analysis as subjects of scientific and philosophical research has not been a lucky one. Thus, *Quasizerlegung* may be considered as a kind of Kuhnian anomaly in the history of epistemology and philosophy of science that has defied philosophical paradigm, be it analytical philosophy proper, logical empiricism, or mainstream philosophy of science. In our opinion, the idea of quasi-analysis has to be seen as a philosophical challenge which is able to blow up the traditional borders of philosophical research. This is all the more true as quasi-analysis, in virtue of its application of mathematical methods and leanings towards problems of *Lebensphilophie* and neutral monism in particular, is a probably unstable mixture of scientifically minded philosophy and irrationalist metaphysical tendencies. This holds, in particular, since mathematics and representation are "protean" concepts that are realized in many different and varying ways (see Mac Lane 1986). The same mathematical structure may have, in fact, many empirical realizations, and representations, in turn, combine technical complexity and overall applicability in a host of different and allegedly divergent scientific and informal contexts. Neutral monism, on its side, runs afoul of philosophies that neatly separate mind and matter as many philosophers from Descartes onwards have done. It puts forward the ingenious hypothesis of bridging the gulf between mind and matter under the aegis of elusive, but still empirically effective, "neutral" entities.

The individual fruitfulness of mathematical representationalism and neutral monism is difficult to be overlooked and underestimated. But what about the Carnapian project of combining them in quasi-analytical framework? As already explained, this is a huge and complex question to answer. We can reasonably claim, however, that the multifaceted character of representation plays a pivotal role here. The accuracy of this submission depends, however, on the idea of representation one subscribes to. Indeed, if one assumes representation as "kopeyliche Betrachtung" that was criticized already by Kant (see Mormann 2018: 3), the marriage of mathematical representation and neutral monism is bound to end in an unhappy and fruitless relation, since the array of entities that the latter needs to represent are epistemically fleeing, or elusive, and thus cognitively unavailable to be copied. On the other hand, things may look brighter, when one shifts to a wider and more flexible account of representation where representing is not a matter of copying but, in a modernized Kantian-style, one of historically intervening and constituting both scientific and ordinary objects (*ibid*.: 5). In this sense, a constitutive and monistic view of mathematical representation may be a good candidate for overcoming restrictive epistemic dichotomies, like the ones already encountered of mind vs. matter and *Leben* vs. *Geist*.

Appendix

The formalism of *Quasizerlegung*: some hints and explanations for the contemporary reader

The formalism of *Quasizerlegung* harks back to the calculus of relation that Augustus De Morgan, Charles Sanders Peirce and Ernst Schröder developed in the second half of the 19th Century (see Givant 2017: 27-8[9]). Aside from theorems (1)-(7) that are formulated in natural language (3-4), it is evidently applied in the discussion of theorems (*Lehrsätze*) (8)-(47) and also invoked in the first section of the manuscript (2), where the author claims to justify an overall applicability of his method, regardless of the basic relations that one assumes at the outset. Let us refer, for simplicity, to binary relations and define them as follows (cf. *Abriss*: 25):

> (D1) A binary relation R on a set X is defined to be a subset of the set $X \times X$ of all ordered pairs $\langle x_1, x_2 \rangle$ of elements x_1, x_2 in X.

There are different ways of visualizing relations. Indeed, one may visualize them by using graphs, matrixes or lists of ordered n-tuples (see *Abriss*: 26-8; Givant 2017: ch. 1, *passim*). As one can clearly see in the text (5 ff.), in *Quasizerlegung* Carnap chose the third way and introduced as an example a similarity structure of the 12 sounds h, l, k, … by listing its positive pairs.

A relation R is said to be *included* in a relation V if and only if every pair of R belongs to V. Following *Abriss* (28) and *Quasizerlegung* (2), inclusion may be symbolically expressed by writing,

$$V \Subset R \, .$$

Relational inclusion corresponds to set-theoretical inclusion, $\subseteq$, (cf. Givant 2017: 2) and satisfies the laws of reflexivity, anti-symmetry and transitivity (see *Abriss*: 29). Also, two relations R and S are defined to be *equal* $V = R$ if and only if both $V \Subset R$ and $R \Subset V$ hold, i.e., if and only if they contain the same ordered pairs. Two special relations are introduced, namely the *identity*

[9] For further historical details see Maddux 2006: 1, *passim*; Maddux 1991.

relation (2), I, and the *diversity relation* ($Abriss$: 26), $\neq$. The former is a reflexive, symmetric and transitive relation and consists of pairs of equal elements. The latter is a symmetric relation and consists of pairs of unequal elements.

There are several constructs for building new relations from already given ones. Suppose that R and S are again relations. The *union* $R \dot{\cup} V$ of R and V (2; *Abriss*: 28-9) is the relation consisting of the pairs that are either in R or in S,

$$R \dot{\cup} V = \{\langle x,y\rangle | \langle x,y\rangle \in R \; or \; \langle x,y\rangle \in V\}$$

The *intersection* $R \dot{\cap} V$ of R and V (2 ff.) is the relation consisting of the pairs that are in both R and V,

$$R \dot{\cap} V = \{\langle x,y\rangle | \langle x,y\rangle \in R \; and \; \langle x,y\rangle \in V\}$$

If R is a relation on a set X, then the *complement* of R, $\dot{-} R$ ($Abriss$: 28), is the relation consisting of the pairs that are in $X \times X$, but not in R. Also, the *difference* of V and R (*ibid.*) is the relation consisting of the pairs that are in R, but not in V,

$$R \dot{-} V \; = \; \{\langle x,y\rangle | \langle x,y\rangle \in R \; and \; \langle x,y\rangle \notin V\}$$

Union, intersection, difference and complement of relations basically correspond to set-theoretical union, $\cup$, intersection, $\cap$, and difference, $-$, and satisfy the related laws of the set-theoretical operations, e.g., associativity, commutativity, distributivity, De Morgan etc. (*ibid.*).

Finally, the *converse*, or *inverse*, of R, $\breve{R}$, (5,6) consists of the pairs in R, but with the reversed order, in symbols,

$$\breve{R} = \{\langle x,y\rangle | \langle y,x\rangle \in R\}$$

It satisfies the law of involution $\breve{\breve{R}} = R$, i.e., the converse of the converse of R is equal to R itself (*Abriss*: 36; Cf. also Maddux 2006: 21).

While the constructs above build new relations from already given ones, the following ones serve, instead, to decompose relations. The *domain* D'R of the relation R consists of all the left-hand members of the pairs in R (7). In symbols,

$$\sqsupset{}'R = \{x | \; for \; some \; y \; \langle x,y\rangle \in R\}$$

Conversely, the *range*, $\sqsubset{}'R$, consists of all the right-hand members of the pairs in R,

$$\sqsubset{}'R = \{y | \; for \; some \; x \; \langle x,y\rangle \in R\}$$

The *field* C'*R* is the set-theoretical union of the two previous sets, C'*R*= ⊃'*R* ∪ ⊂ '*R*. For further information about the formal properties of these operators and alternative symbolic expressions, consult *Abriss* (35-8) and Givant 2017 (14, 148). Among the methods of decomposing constructions, one may also characterize the *projection* $\vec{R}'y$ which maps each element of the field C'*R* to the set of its left-side companions in *R*. In symbols,

$$\vec{R}'y = \{x|\langle x, y\rangle \in R\}$$

Mutatis mutandis, one can also characterize the projection $\overleftarrow{R}'x$ (see *Abriss*: 35). Projections appear very often throughout the manuscript (5-6 ff.). More specifically, Carnap uses their respective specular operators to define the concept of similarity neighbourhood (5). Obviously, whenever the relation *R* is, such as similarity, a symmetric relation (*ibid.*), $\vec{R}'x$ and $\overleftarrow{R}'$x result in the same outputs. As a consequence of this, while choosing to work with similarity as basic relation, the author does not tell the difference between the two operators in the text.

Projections can be also defined as functions that map C'*R* to its powerset. Following the calculus of classes, both in *Abriss* (25) and *Quasizerlegung* (10) Carnap indicates the powerset of a set *X* by Cl'*X*. Abstraction operators and union/intersection of families of sets are correspondingly expressed by capped variables without brackets followed by abstraction conditions, $\hat{x}$ $(\dots x \ \dots)$ (cf. for instance 5, 7, definitions (8) and (16)) and apostrophized letters " s' ", " p' " followed by the family of sets they are wanted to unify or intersect (cf. for instance 9, 10, theorems (25) and (34)). These notations are quite different from the more recent one, $\{x| \ \dots \ x \ \dots\}$, $\cup$, $\cap$, and go back to *Principia Mathematica* (see also Marciszewski 1981).

Two further methods of constructions can now be introduced, namely *relative product* and *restriction*.

The *relative product*, or *(relational) composition*, *R*|*V* (2; *Abriss*: 38-9) is the relation consisting of the pairs $\langle x, z\rangle$ such that for some y if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in V$, then $\langle x, z\rangle \in R|V$ (see also Maddux 2006: 7 and Givant 2017: 6). It satisfies the law of associativity, i.e., $(R|V)|T = R|(V|T)$ (see *Abriss*: 39 for instance). The relational product of *R* with itself, *R*|*R*, is denoted by R^2. As is easily seen the relation *R* is a transitive relation if and only if $R^2 \subseteq R$ (see *ibid.*: 40).

In *Abriss* (37-8), there are several types of relation *restrictions*, like *domain-restriction* or *range-restriction*, expressed by operators ↾, ↿. For simplicity, we will define only range-restriction, which is also the only one that *Quasizerlegung* applies. In particular, if $\beta \subseteq$ ⊂ 'R, then $R \upharpoonright \beta$ is the set of the *R*-pairs whose right-hand members are contained in β. In symbols,

$$R \upharpoonright \beta = \{\langle x, y\rangle | \langle x, y\rangle \in R \; and \; y \in \beta\}$$

Through restrictions, another special relation can be defined, which consists of the pairs of equal elements on the field of R. It is introduced in the first section of *Quasizerlegung*, where Carnap indicates it by $I \upharpoonright C'R$ (2). The same relation can also be expressed by R^0 (see *Abriss*: 37-8).

Finally let us mention the *iterated relative product* of a relation with itself (2). It is indicated by R_* and consists of the following union of relations:

$$R_* = R^0 \dot{\cup} R \; \dot{\cup} \; R^2 \dot{\cup} R^3 \dot{\cup} \ldots$$

R_* is defined to be a *R-chain* if, whenever $< x, y >, < y, z > \in R_*$, we have $< x, z > \in R_*$ for all $x, y, z \in C'R$. In case *R* is infinite one seems to need the axiom of choice or some similar principle to ensure the existence of R_*. For further formal properties of *R*-chains, consults *Abriss* (56-7).

It might be useful to conclude this section by briefly clarifying the meanings of the logical and auxiliary symbols that are applied in the manuscript. Concerning the logical symbols, Carnap uses the symbols $\supset, ., \vee, -, \equiv$ to respectively indicate connectives $\rightarrow, \wedge, \vee, \neg, \leftrightarrow$. Moreover, the operators $\exists x, (x)$ and ɿ stand for existential, universal and definite description quantifiers (see *Abriss*: chs. 4-7). Concerning the auxiliary symbols, following PM, Carnap used a kind of "point calculus" for round, square, and curly brackets that use to be employed in modern treatises. One can find the rules of their use in *Abriss* (9-10). Depending on the nesting level, more dots, e.g. :, ::, or :., stand for square or curly brackets. Dots lie beside connectives $\supset$, $\vee$ and $\equiv$, and follow both the symbol of deduction "$\vdash$" and quantifiers. Every point brings together symbols either to the end or to the next points.

We conclude this appendix putting together several definitions of "similarity circles" (6 ff.), or *Ähnlichkeitskreise*. Similarity circles are paramount in quasi-analysis, for they are the building blocks, or prime elements, (cf. § 3 above) of similarity structures[10]. They are also the nodal point of Goodman 1951 harsh criticism of unfaithfulness and inaccuracy against quasi-analysis (see Goodman 1951: 157-161; cf. Leitgeb 2007: 193-200 ff.). Carnap introduces similarity circles in *Quasizerlegung*, *Abriss* and *Aufbau*. Basically, their introductions display formal or informal language and differ in applying the constructs of relation algebra or not. For a set to be a similarity circle two fundamental conditions must be met, 1) that of being a homogeneous similarity structure (i.e. every element is similar to every other) and 2) that of being

[10] Some interesting non-trivial results of the theory of similarity circles can be found in Brokhaus (1963). Discussed in Mormann (2007).

trivially included in other similarity circles (i.e. if A and B are similarity circles and $A \subseteq B$, then $A = B$). Before defining similarity circles, the author introduces similarity relations. To this end, he explicitly employs the above-mentioned constructs of relation algebra only in *Aufbau*, where similarity, S, is defined by applying the relational operators of union $\dot{\cup}$, conversion $\breve{}$ and identity ^{0} to a given asymmetric relation R: $S = R \mathbin{\dot{\cup}} \breve{R} \mathbin{\dot{\cup}} R^{0}$ (see *Aufbau*: 179). On the other hand, *Quasizerlegung* (2), as well as Mormann (2009: 255), introduce similarity invoking symmetry and reflexivity axiomatically. Following *Aufbau* (§80), similarity circles are informally defined in the following way:

> (D2): A similarity circle, *SC*, is a subset of elements of a similarity structure such that: (i) any two elements of *SC* are similar one to the other; (ii) if an element is similar to all elements of *SC*, then it belongs to it.

The conditions (i) and (ii) express, respectively the above-mentioned informal conditions of homogeneity 1) and maximality 2). Formal definitions of similarity circle present different nuances in *Quasizerlegung* and *Abriss* in accordance with the author's choice of using the constructs of relation algebra or not. Let us begin with the definition provided by Mormann (2009: 259), which is more familiar to the reader:

> (D3) Let $(S, \sim)$ be a similarity structure. A subset T of S is a similarity circle (*Ähnlichkeitskreis*) of $(S, \sim)$ iff it is a maximal set of similar elements, i.e., iff it satisfies the following two conditions: (I) For all x, y $(x, y \in T \Rightarrow x \sim y)$; (II) For all $z \in S$ $(z \sim x$ for all $x \in T \Rightarrow z \in T)$.

As can be easily seen, conditions (I) and (II) of (D3) are but the formal version of conditions (i) and (ii) of (D2). As we shall see in a moment, (D3) strictly reflects the introduction of similarity circles in *Quasizerlegung* (6, Definition (13)), see below (D4). Differences between (D3) and (D4) lie in logical notations, calculus of classes with capped second order variables and punctuation (., :, :: etc.).

> (D4) We define the class of the <u>similarity circles</u> as follows: $sim = \hat{\beta}\ (x, y\ \in\ \beta\ \supset xSy \mathrel{:.} (z) \colon (u). u \in \beta\ \supset uSz. \supset. z \in \beta)$ Def.

The first and second parts of (D4), namely $x, y\ \in\ \beta\ \supset xSy$ and $(z) \colon (u). u \in \beta\ \supset uSz. \supset. z \in \beta)$ respectively correspond to (i) and (ii). As can be appreciated, both definitions (D3) and (D4) employ predicative logic without resorting to the constructs of relation algebra. On the other hand, they clearly figure in the definition, here called (D5), given in *Abriss* (49):

(D5) SC = Df $\hat{\alpha}\ (\alpha \upharpoonright \alpha \Subset R :. (x): \alpha \subset \vec{R}'x \supset x \in \alpha)$

The definition (D5) uses constructs of relational algebra and assumes that *R* is a similarity relation. The first part of the definition, i.e., $\alpha \upharpoonright \alpha \Subset R$, ensure *SC* satisfies the condition 1) of homogeneous similarity structure, while the second part, i.e., $(x): \alpha \subset \vec{R}'x \supset x \in \alpha)$, ensures 2), that is, *SC* to be maximal. The constructs of relation algebra here employed are those of double restriction, $\alpha \upharpoonright \alpha$ and projection, $\vec{R}'x$. The double restriction, which is not numbered among in the operators listed above, is defined in *Abriss* (38) and amounts to the product of α and itself, namely the set $\alpha \times \alpha$ of all ordered pairs $\langle x_1, x_2 \rangle$ of elements x_1, x_2 in α. Projection is used, instead, to define similarity neighbourhood in *Quasizerlegung* (5) and is of some importance in contemporary debate. As noticed by Carnap (6), constructing a quasi-analysis on the base of similarity neighbourhoods is not possible. They allow, however, as Mormann 2009 (269 ff.) has shown, to find new order structures from given similarity ones and, as a result, to defeat criticisms of similarity structure for being too weak to be useful (cf. Goodman 1951; Quine 1969). By the way, the possibility of generating an asymmetric relation from similarity through similarity neighbourhoods was already detected in *Quasizerlegung*, when Carnap applied the projection $\vec{S}'$ to define the relation xE_1y, which holds whenever the similarity neighbourhood of *x* is a subset of the neighbourhood of *y* (5).

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